Does Ambiguity Matter?  
Estimating Asset Pricing Models with a Multiple-Priors Recursive Utility

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Abstract

This paper considers continuous time asset pricing models with stochastic differential utility incorporating decision makers’ concern with ambiguity on true probability measure. In order to identify and estimate key parameters in the models, we use a novel econometric methodology developed recently by Park (2008) for the statistical inference on continuous time conditional mean models. The methodology only imposes the condition that the pricing error is a continuous martingale to achieve identification and obtain consistent and asymptotically normal estimates of the unknown parameters. Under a representative agent setting, we empirically evaluate alternative preference specifications including a multiple-prior recursive utility. Our empirical findings are summarized as follows: Relative risk aversion is estimated around 1.5-5.5 with ambiguity aversion and 6-14 without ambiguity aversion. Related, the estimated ambiguity aversion is both economically and statistically significant and including the ambiguity aversion clearly lowers relative risk aversion. The elasticity of intertemporal substitution (EIS) is higher than 1 but the identification of EIS appears to be fairly weak, as observed by many previous authors, though other aspects of our empirical results seem quite robust.

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1 Introduction

In this paper, we examine continuous time asset pricing models with recursive preferences incorporating decision makers’ concern with ambiguity on true probability measure. Our measure of ambiguity aversion is derived using some properties of continuous time diffusion processes and conditional volatilities of asset returns play an important role in quantifying premiums driven by ambiguity aversion. Meanwhile the pricing kernel of the model is set as the intertemporal marginal rate of substitution of a representative agent and identifying this kernel involves macroeconomic variables such as consumption growth, which are available only at lower frequencies. Consequently we need to handle both financial data sampled at a high frequency and macroeconomic data available only at lower frequencies in order to estimate these asset pricing models. This motivates us to develop a new set of econometric tools for the statistical inference on continuous time conditional mean models when available data series are of mixed frequencies. With this method in hand, we measure the extent to which financial markets price ambiguity, risk, and intertemporal substitutability.

Since the seminal papers by Hansen and Singleton (1982) and Mehra and Prescott (1985), a large body of work has sought after more relevant forms of the preferences of economic agents to explain asset market behaviors. The main reason for this direction of the study is because time-separable expected utility functions equipped with a constant relative risk aversion (CRRA) impose a potentially restrictive relationship between the risk aversion and intertemporal substitution. Under power utility models, for instance, the elasticity of intertemporal substitution (EIS) is given by the reciprocal of the coefficient of relative risk aversion, which may result in various complications such as equity premium, volatility and interest rate puzzles. Epstein and Zin (1989, 1991) investigated an important generalization of the standard power utility model by considering a class of recursive utility functions. They provide a theoretical framework in which the agent can have distinct attitudes toward intertemporal substitution and risk. This flexibility may offer a possible solution for various asset price anomalies because a high (low) risk aversion does not necessarily imply a low (high) elasticity of intertemporal substitution.

A caveat is that it is difficult to model the preference toward Knightian uncertainty or ambiguity within the original recursive utility framework due to the assumption of single prior held by investors. However the Ellsberg paradox suggests that decision makers prefer an unambiguous situation, other things being equal. In response to this, Gilboa and Schmeidler (1989) built a multiple-priors model to incorporate ambiguity aversion in an atemporal setting. Epstein and Wang (1994) develop a dynamic version of Gilboa and Schmeidler in
a discrete-time framework and Epstein and Schneider (2003) provide axiomatic foundations for recursive multiple-priors utility. Chen and Epstein (2002) focused on the formulation of utility in continuous time that allows a distinction between risk aversion and ambiguity aversion, as well as the distinction from the EIS. In order to handle the additional dimension of preferences, they extended the continuous time version of the recursive utility (stochastic differential utility) proposed by Duffie and Epstein (1992) such that the model includes a set of priors rather than a single prior. According to Chen and Epstein (2002), the economic agents will have multiple prior beliefs on the state of the nature and they form a set of expectations based on their beliefs. Due to the fact that fundamental shock processes are generated by Brownian motion, the degree of ambiguity is described by an additional term distorting the conditional mean component of the implied asset return processes and the decision maker chooses a probability measure using the maxmin principle following Gilboa and Schmeidler (1989).8

Despite the appealing features of the multiple-priors recursive utility model, there has been little econometric work on estimating the model compared to other utility specifications. To the best of our knowledge, this is the first paper that empirically tackles this issue under the framework of consumption-based models. The multiple-priors recursive utility model has a multi-factor beta representation of asset returns; (i) covariance between returns and consumption growth, (ii) covariance between returns and aggregate wealth return, and (iii) covariance between returns and ambiguity. However this structure makes identification of the model difficult because aggregate wealth and volatility of returns are unobservable latent variables and more notably, there is a lack of econometric methodology for estimating continuous time models. Below, we briefly explain how we overcome these issues.

With regard to the unobservability of aggregate wealth, several approaches have been suggested. The baseline approach would be to use a market return as a proxy for the returns on aggregate wealth (e.g., Epstein and Zin (1991), Bakshi and Naka (1997) and Normandin and St-Amour (1998)). However, the aggregate wealth portfolio should be a broader measure than the financial market portfolio, because the former includes human capital and housing wealth etc. as well as the financial wealth. Therefore the market return only covers a subset of the aggregate wealth returns. Another approach is to use a specific structure for the unobservable wealth by incorporating the dynamics of consumption growth and utility continuation value. Given the imposed structure, the aggregate wealth is implicitly given by consumption and utility continuation value. Therefore, this approach enables them to replace the unobservable wealth return with the specific structure imposed on the consumption and the future utilities. Chen et al. (2008) exploit the Euler equation to estimate future continuation utility in a non-parametric way.

Although this method is attractive, it is difficult to use in our continuous-time framework handling mixed frequencies of data. Instead, we consider a different approach to overcome the difficulties from the unobservable aggregate wealth. The aggregate wealth return is

\[ \max_{c} \min_{Q \in P} E^{Q}[u(c)] \]

8There exists a related line of work on robust decision making. Hansen and Sargent (2001) and their co-authors emphasize ‘model uncertainty’ and the concern on the misspecification, which is similar in spirit to ambiguity aversion à la Gilboa and Schmeidler.
a return on the claim which gives a stream of future consumption. In this sense, the consumption of each period is financed by the aggregate wealth return, and therefore, we can think of the aggregate wealth as the sum of financial wealth and human capital, which are the two largest sources of the income in an economy. Thus, the unobservability of aggregate wealth falls mostly on the human wealth. Following Campbell (1993), we assume that the proportion of the financial wealth to the human wealth is stationary and moreover, the labor income is homogeneous of degree one with respect to the human wealth. In this case, the unobservable wealth can be substituted by a linear combination of market return and labor income growth. This simple structure makes the asset pricing formula tractable so that we can directly compare the results of alternative models.

For an empirical analysis of our model, we use the martingale regression method recently developed by Park (2008) for inference on continuous time conditional mean models. It identifies the true parameter value simply by imposing the martingale condition for pricing error, utilizing the fact that the conditional expectation of pricing error is zero for the true parameter value, whereas it is generally non-zero for other values of parameter in the pricing equation. The spirit of the methodology is therefore somewhat similar to the GMM estimation for the nonlinear Euler equation models (e.g. Hansen and Singleton (1982)). The martingale condition for pricing error can easily be handled, since any continuous martingale can be converted into Brownian motion due to the celebrated theorem by Dambis, Dubins and Schwarz. The theorem states that any continuous martingale becomes Brownian motion if it is read after time change defined by the generalized inverse of its quadratic variation. The actual martingale estimator is defined as a minimum distance estimator based on the discrepancy between the empirical distribution of normalized pricing errors after time change and the standard normal distribution.

There are several attractive features of our approach using the martingale estimation. First, the martingale estimation does not require any parametric specification of volatilities in pricing errors. However, it allows for, and is robust with respect to, the presence of a wide variety of both deterministic and stochastic volatilities in pricing errors. This is an important advantage, since many empirical researches on the financial data find strong evidences that stock returns possess time-varying and potentially stochastic volatilities, while the exact natures of these volatilities are difficult to specify more precisely. Second, the martingale estimation does not use the orthogonality condition to identify the true parameters. Instead, it only imposes the martingale condition for pricing errors, and subsequently uses the time change theorem by Dambis, Dubins and Schwarz to make the condition implementable in defining the martingale estimator. Unlike the GMM estimator, we do not need instruments to compute the martingale estimator. Yet, the martingale estimator naturally accommodates endogeneity, and it is free of any kind of endogeneity problem.

Last but not least, this method allows applied econometricians to directly tackle asset pricing models formulated in continuous time. Many asset pricing models are developed in continuous time partly because of its mathematical elegance and tractability. However, it is also true that continuous time models give better descriptions for many financial markets, which clear at very high frequencies. Choosing an empirical model to fit at a most relevant frequency will certainly reduce the possibility of data missaggregation bias and decision bias. As mentioned earlier, however, macro variables are only observed at lower frequen-
cies. Therefore, we have to deal with mixed frequencies of data in models involving both macro and financial variables. The martingale method provides an effective solution to this problem. The observations on financial variables at high frequencies are used to identify the model and also to nonparametrically correct for volatilities in pricing errors and after identification and volatility correction, the model is estimated by observations sampled at random intervals of lower frequencies.

Using daily data on asset returns and monthly and quarterly macroeconomic data from 1960 to 2006, we estimate several specifications of recursive utility framework. According to our results, the estimates of ambiguity aversion parameter are both economically and statistically significant. This is a highly robust feature of the data and the estimates are almost invariant to specifications. In addition, relative risk aversion is estimated around 1.5-5.5 with ambiguity aversion and 6-14 without ambiguity aversion. That is, the ambiguity aversion lowers the estimates of the relative risk aversion in all cases we have considered. Suppose investors receive information from stock prices which may include some noisy signals. If those signals are hard to interpret and hence difficult to extract fundamentals, they would prefer an asset market with less ambiguous information flows and request premiums for bearing such uncertainty different from risk in a Knightian sense. Given that, our empirical results suggest that risk aversion parameter can have an upward bias sans an adjustment for ambiguity aversion to account for high average market returns.

Another important preference parameter is the elasticity of intertemporal substitution (EIS). Recently, estimating the EIS has drawn much attention and existing studies report a wide range of values including even negative numbers. According to our estimations, the EIS is higher than 1; specifically 1.3-22 with ambiguity aversion, and quite high without ambiguity aversion. We find that values of the objective function of our minimum distance estimator measured by the Cramer-von Mises statistic is very flat around the values of the reciprocal of the EIS between 0 and 3, meaning that the EIS may be observed in a wide range between a number close to 0 and a very large positive number. Based on extensive robustness checks, we argue that the weak identification issue of the EIS parameter results from the combination of smooth variations of consumption growth and parametric restrictions imposed in preferences. One notable finding is that EIS estimation appears to be tighter when ambiguity aversion is incorporated and this result is robust to alternative specifications.

The remainder of the paper begins with describing our theoretical model in Section 2. For comparison, we also consider other baseline models, which can be considered as special cases of our model. Section 3 accounts for the statistical underpinnings of our econometric methodologies. Section 4 explains our empirical procedure and resultant measurements necessary for our empirical evaluation. Section 5 shows and discusses our main results. Then we conclude in Section 6.

2 A Recursive Utility Model with Ambiguity Aversion

Consider a probability space \((\Omega, \mathcal{F}, P)\) which describes the uncertain nature of the economy. Define a standard one dimensional Brownian motion \((W_t)\) on \((\Omega, \mathcal{F}, P)\) and the Brownian
filtration \( (\mathcal{F}_t)_{0 \leq t \leq T} \), where \( \mathcal{F}_t \) is the \( \sigma \)-field generated by \( (W_s)_{s \leq t} \). The time horizon \([0, T]\), where \( T \) is finite. Suppose that the representative decision maker does not know the true probability measure and has to choose a subjective probability measure from the set of all priors \( \mathcal{P} \) which are uniformly absolutely continuous with respect to the true \( P \) in \( \mathcal{P} \).\(^9\) Duffie and Epstein (1992) show that for a fixed consumption process \( C \) and a probability measure \( Q \in \mathcal{P} \), there exists a utility process \( V_t^Q \) uniquely solving

\[
V_t^Q = \mathbb{E}^Q \left[ \int_t^T f(C_s, V_s^Q) ds \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T,
\]

where \( \mathbb{E}^Q(\cdot|\mathcal{F}_t) \) is the conditional expectation operator and \( f(C, V) \) is called a normalized aggregator function linking current consumption and the future value.\(^{10}\)

From now on, we use the functional form

\[
f(C, V) = \frac{C^{1-\beta} - \phi(\alpha V)^{1-\beta}}{(1-\beta)(\alpha V)^{1-\beta}}
\]

for some \( \phi \geq 0, \beta \neq 1, \alpha \leq 1 \). This can be regarded as the continuous-time version of Kreps-Porteus utility function in which \( \alpha \) and \( \beta \) measure the degree of relative risk aversion (RRA) and the elasticity of intertemporal substitution (EIS) respectively. Specifically, the RRA is measured by \( (1-\alpha) \), and the EIS is \( 1/\beta \). In addition, following Epstein and Zin (1989), relative sizes of these two measures are related to the investor’s attitude toward the speed of resolving uncertainty: If the RRA \( (1-\alpha) \) is larger (smaller) than the reciprocal of the EIS \( (\beta) \), the investor prefers an early (a late) resolution of uncertainty. The additional feature of this model compared to the conventional recursive utility model is that the consumer chooses a probability measure from available priors, which justifies the name, ‘multiple-priors utility’. Under this extra layer of uncertainty which leads to the Ellsberg paradox, Gilboa and Schmeidler (1989) suggested the following minimax type of value function

\[
V_t = \min_{Q \in \mathcal{P}} V_t^Q, \quad 0 \leq t \leq T.
\]

The multiple-priors recursive utility is given by the lower envelope of the utility process \( (V_t^Q) \) which is determined by the conditional expectation of future consumption and utility values. Chen and Epstein (2002) showed that there exists a unique solution to (3) satisfying the dynamic consistency under certain conditions.\(^{11}\) As clearly seen from (??), the Girsanov

\[^9\] \( \mathcal{P} \) is uniformly absolutely continuous with respect to \( P \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( E \in \mathcal{F} \) and \( P(E) < \delta \) imply \( Q(E) < \varepsilon \), for all \( Q \in \mathcal{P} \).

\[^{10}\] From the martingale representation theorem, we can express (1) in a differential form of

\[
dV_t^Q = -f(C_t, V_t^Q) dt + \sigma_t^Q dW_t^Q,
\]

where \( V_T^Q = 0 \), \( (W_t^Q) \) is the standard Brownian motion under \( Q \)-measure, and \( \sigma_t^Q \) is endogenously determined.

\[^{11}\] Dynamic consistency in this paper is defined in the following sense. If two consumption plans \( c \) and \( c' \) are the same up to a stopping time \( \tau \), and the value \( V_{\tau} \) of \( c \) is weakly preferred to that of \( c' \) at \( \tau \) almost surely, then \( V_0(c) \geq V_0(c') \) almost surely with a strict inequality in the case that \( P(V_{\tau}(c) > V_{\tau}(c')) > 0 \) holds.
transformation lies at the heart of constructing a set of priors $P$ on $(\Omega, \mathcal{F}_T)$. Specifically, they define a density generator $(\lambda_t)$ for which the process $(z_t^\lambda)$ is a $P$-martingale, where

$$dz_t^\lambda = -z_t^\lambda \lambda_t dW_t, \quad z_0^\lambda = 1,$$

or equivalently,

$$z_t^\lambda \equiv \exp \left( -\frac{1}{2} \int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s \right), \quad 0 \leq t \leq T.$$ 

Then, $(z_t^\lambda)$ is set as the Radon-Nikodym derivative $dQ/dP$ on $(\mathcal{F}_t)$ and $P$ is defined as the set of $Q$-measures produced by the density generator. We assume that the Novikov condition holds to suffice the existence of such $(\lambda_t)$. Since all the prior beliefs are absolutely continuous with $P$, we can expect from the Girsanov’s theorem that any subjective utility $(V^Q_t)$ given an equivalent measure $Q \in P$ will modify the drift function of the utility continuation process by $(\lambda_t \sigma_v)$. This is because $(W_t)$ is the Brownian motion under $P$ measure, but not under $Q$. That is, by shifting $(\lambda_t)$, we can generate a continuum of subjective utility functions differing in terms of probability distribution within the class of absolutely continuous multiple-priors. Chen and Epstein (2002) showed that the differential form of (3) is

$$dV_t = \left\{ -f(C_t, V_t) + \max_{\lambda_t \in \mathcal{L}} \lambda_t \sigma_v \right\} dt + \sigma_v dW_t, \quad (4)$$

where $\mathcal{L}$ is to be defined in short.

To further analyze the additional term in (4), assume that $(\lambda_t)$ is bounded by some constant $\kappa > 0$. This makes the domain of $(\lambda_t)$ defined as

$$\mathcal{L} = \{(\lambda_t) : \sup\{|\lambda_t| : 0 \leq t \leq T\} \leq \kappa\}.$$ 

That is, the subjective beliefs have some boundary defined by a constant $\kappa$. We can interpret the multiple priors as the subjective beliefs for which the worst case scenario of the economic agents is confined by the case defined by $\kappa$. Hereafter, we examine the multiple-priors model with a boundary restriction for $(\lambda_t)$ with $\kappa > 0$.\footnote{Chen and Epstein (2002) call this specification “$\kappa$-ignorance” case. This is closely related to the condition guaranteeing the dynamic consistency, called rectangularity. For a nice interpretation on the rectangularity condition and dynamic consistency, see Epstein and Schneider (2003).}

Under the standard environment of the economy, first order conditions for optimal consumption choice can be expressed in terms of the supergradient of utility at the optimal consumption $C$.\footnote{A supergradient for $V$ at $C$ is a process $(\Lambda_t)$ with $\mathbb{E} \left\{ \int_0^T \Lambda_t \cdot (C_t' - C_t) dt \right\} \geq V(C') - V(C)$ for all admissible $C'$. For more details, see Duffie and Skiadas (1994) and Chen and Epstein (2002).} Especially, $C$ is optimal if

$$\Lambda_t = \exp \left\{ \int_0^t f_v(C_s, V_s) ds \right\} f_c(C_t, V_t) z_t^{\lambda^*_t}, \quad \text{for all } t, \quad (5)$$

where $(\Lambda_t)$ is the state-price process (or intertemporal marginal rate of substitution process, IMRS) and $(\lambda^*_t)$ is the maximizer of the ambiguity compensation $(\lambda_t \sigma_v)$ for any given $(\lambda_t)$.
such that $|\lambda_t| \leq \kappa$ for all $0 \leq t \leq T$. Then the IMRS in our case is given as

$$\Lambda_t = \exp \left\{ \int_0^t \left( -\phi + \frac{\phi - 1 - \beta}{1 - \beta} \left( C_s^{(1-\beta)}(1) - (\alpha V_s)^{(1-\beta)/\alpha} \right) ds \right\} \phi C_t^{-\beta}(\alpha V_t)^{\beta/\alpha} z_t^\kappa. $$

Using Ito’s lemma and no arbitrage principle, we can show that, for an asset $i$,

$$\frac{dp_i^t}{p_i^t} - r_i^t dt = \mathbb{E} \left( \frac{dp_i^t}{p_i^t} \frac{d\Lambda_t}{\Lambda_t} | \mathcal{F}_t \right) + \sigma_i^t dW_t$$

(6)

$$= \left\{ \frac{\beta \alpha}{1 - \beta} \rho_y^i \sigma_t^m \sigma_t^h + \left( 1 - \frac{\alpha}{1 - \beta} \right) \rho_h^i \sigma_t^m \sigma_t^h + \kappa \sigma_t^i \right\} dt + \sigma_i^t dW_t,$$

where $(\sigma_t^i)$ is the volatility of aggregate wealth $(G_t)$ for which the return is given by

$$dr_t^q = \mu_t^q dt + \sigma_t^q dW_t, \quad dG_t = G_t(dr_t^q - C_t dt),$$

$(\sigma_t^q)$ is the volatility of consumption growth, $\rho_y^i$ and $\rho_h^i$ are the correlation coefficients of consumption growth and aggregate wealth return with the return of an asset $i$, respectively.

Equation (6) is a three-factor CAPM of the cross-sectional asset pricing model; the risk premium of any tradable asset $i$ with return $(dp_i^t/p_i^t)$ is determined by the covariance between returns and consumption growth, covariance between returns and aggregate wealth, and covariance between returns and density generator. Notice that the standard CRRA utility specification, such as power utility, only has the first factor, while the single-prior recursive utility models (e.g. Epstein and Zin (1989, 1991) and Duffie and Epstein (1992)) have the first two factors.

In order to measure the aggregate wealth, we assume the wealth process $(G_t)$ has two components - financial wealth $(M_t)$ and human wealth $(H_t)$,

$$G_t = M_t + H_t.$$ 

(7)

From Ito’s lemma we have

$$\sigma_t^g = \sqrt{\pi_t^2 (\sigma_t^m)^2 + (1 - \pi_t)^2 (\sigma_t^h)^2 + 2 \rho_h \sigma_t^m \sigma_t^h},$$

(8)

where $\pi_t = M_t/G_t$ is the proportion of financial wealth to the total wealth at time $t$, $(\sigma_t^g)$ and $(\sigma_t^h)$ are the respective diffusion coefficient of $(dM_t/M_t)$ and $(dH_t/H_t)$, and $\rho_h$ is the correlation coefficient between market return and labor income growth. In particular, we specify the human capital process by

$$dH_t = H_t(dr_t^h) - Y_t dt, \quad ds_t^h = \mu_t^h dt + \sigma_t^h dW_t, $$

(9)

where $Y_t$ is real labor income at time $t$. Note that the labor income $Y_t$ is financed from the return on the human capital $H_t$ at time $t$. From (8) and (9), the covariance between the aggregate wealth and individual return for an asset $i$ is written as

$$\rho_y^i \sigma_t^g \sigma_t^i = (\pi_t \sigma_t^m + (1 - \pi_t) \rho_h^i \sigma_t^h) \sigma_t^i.$$
For simplicity, we assume that $\pi_t = \pi$ for all $t$. This is true, for instance, under the steady state of the economy, in which the proportion of aggregate wealth to the financial wealth is constant over time. Moreover, we assume that the labor income is homogeneous of degree one with respect to human capital, especially, $Y_t = \psi H_t$ for all $t$ with some constant $\psi$.

Given our interest in the behavior of the market return, we set $i = m$, and therefore the fundamental asset pricing equation of the multiple-priors recursive utility model is expressed as

$$\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta \alpha}{1 - \beta} \rho_c \sigma_c^m \sigma_t^m dt + \left(1 - \frac{\alpha}{1 - \beta}\right)(\sigma_t^m)^2 \pi dt + \left(1 - \frac{\alpha}{1 - \beta}\right) \rho_y \sigma_t^m \sigma_y^m (1 - \pi) dt + \kappa \sigma_t^m dt + \sigma_t^m dW_t,$$

(10)

where $(p_t)$ is the price of the market index, $(\sigma_t^m)$ is the instantaneous conditional volatility of labor income growth, and $\rho_c$, $\rho_y$ are the correlation coefficients of consumption growth and labor income growth with the market return, respectively. From now on, we turn our attention to estimating the key preference parameters ($\alpha$, $\beta$, $\kappa$) in (10).

Note that (10) nests many popular asset pricing models as special cases. By imposing a priori restrictions to (10), we can estimate different models to compare the common set of parameters. In fact, we estimate four models from (10), two of which use only the financial wealth with and without ambiguity and the other two models explicitly consider human capital. One important observation from our empirical setting is that time-varying volatilities of macroeconomic variables and asset returns play key roles in both the conditional mean (drift) part and the error (diffusion) terms. Given the ample evidence that those volatilities are highly persistent, this makes identification of the models statistically challenging because of heteroskedasticity, endogeneity, and measurement problems. In addition, the equilibrium relationship (10) that continuously holds need to be properly treated for correct empirical evaluations with discretely sampled data points. In the below, we tackle those issues.

3 Econometric Methodology

3.1 Martingale Estimation of Asset Pricing Models in Continuous Time

Here we explain how to specify and estimate our model (10). Tentatively, we assume that the volatility processes $(\sigma_t^m)$, $(\sigma_t^c)$ and $(\sigma_t^y)$ are observed. In the next subsection, we will explain in detail how we may extract these processes. Moreover, we will set the correlation coefficients $\rho_c$ and $\rho_y$ of consumption and labor income growths with market returns, as well as the fraction $\pi$ of financial wealth, to be known and constants.\textsuperscript{14} These parameters will be calibrated using the values obtained or often assumed in the empirical literature. In what follows, we assume that $(\sigma_t^m)$, $(\sigma_t^c)$ and $(\sigma_t^y)$ are non-constant and time-varying, and that $\rho_c$ and $\rho_y$ are non-zero. These assumptions are necessary for the identification of our model.

\textsuperscript{14}This does not imply that the correlation between the aggregate wealth and market returns are constant. As shown above, it still varies over time.
Now we let $\theta = (\alpha, \beta, \kappa)$ be the vector parameters in our model with the true value $\theta_0 = (\alpha_0, \beta_0, \kappa_0)$, and define $(\Lambda_t(\theta))$ to be the state-price deflator (or IMRS) that is given by

$$
\Lambda_t(\theta) = \frac{\beta \alpha}{1 - \beta} \rho_c \sigma_t^m + \left(1 - \frac{\alpha}{1 - \beta}\right) \{\pi \sigma_t^m + (1 - \pi) \rho_y \sigma_t^y + \kappa\} \sigma_t^m.
$$

(11)

Subsequently, we define the pricing error process $(Z_t(\theta))$ from our model as

$$
dZ_t(\theta) = \frac{dp_t}{pt} - r_f^t dt - \Lambda_t(\theta) dt,
$$

and write

$$
Z_t(\theta) = A_t(\theta) + U_t,
$$

(12)

where $dA_t = -\{\Lambda_t(\theta) - \Lambda_t(\theta_0)\} dt$ and $dU_t = \sigma_t^m dW_t$.

It is clear that the pricing error process $(Z_t(\theta))$ is a semimartingale with the bounded variation component $(A_t(\theta))$ and the martingale component $(U_t)$. Note in particular that $(U_t)$ is a continuous martingale with respect to the filtration $(\mathcal{F}_t)$, to which the Brownian motion $(W_t)$ is adapted. Furthermore, the bounded variation component $(A_t(\theta))$ vanishes if and only if $\theta = \theta_0$ holds under the trivial identification conditions introduced above. Therefore, we may conclude that the pricing error process $(Z_t(\theta))$ becomes a continuous martingale if and only if $\theta = \theta_0$.

Recently, Park (2008) developed a general methodology to estimate and test the continuous-time conditional mean model that is identified by this type of martingale condition for the error process. Below we explain how we can implement his methodology to estimate the unknown parameter $\theta$ in our model. The methodology relies on the celebrated theorem by Dambis, Dubins and Schwarz, which will be referred to the DDS theorem throughout the paper. To introduce the DDS theorem, we denote by $([U]_t)$ the quadratic variation of $(U_t)$, which is given by

$$
[U]_t = \lim_{|t_k| \to 0} \sum_k (U_{t_k} - U_{t_{k-1}})^2,
$$

where $|t_k|$ is the mesh of partition $(t_k)$ of the interval $[0, t]$. We assume that $[U]_t \to \infty$ a.s. as $t \to \infty$. Moreover, we introduce the time change $(T_t)$, which is defined as

$$
T_t = \inf\{s \geq 0 | [U]_s > t\}.
$$

(13)

The DDS theorem says that if $(U_t)$ is a continuous martingale, then there exists a standard Brownian motion $B$ such that $U_t = B_{[U]_t}$, or equivalently,

$$
U_{T_t} = B_t.
$$

---

15 As can be clearly seen, we may identify up to four unknown parameters in our model. Therefore, for instance, we may regard $\pi$ as unknown and estimate it as an additional unknown parameter. However, the estimate for $\pi$ is unstable and unreliable.

16 We temporarily assume that there is no jump in the pricing error process to focus on the main idea of the methodology. Indeed, it can be applied to the processes with jumps with some simple modifications, which we will explain later in this subsection.
The Brownian motion \( B \) is called the DDS Brownian motion of \( U \). See, e.g., Revuz and Yor (2005) for the proof and more discussions about the DDS theorem. In most applications, \( ([U]_t) \) is strictly increasing, in which case \( T \) is just the time inverse of \( ([U]_t) \). Roughly, the DDS theorem implies that if we read a continuous martingale using a clock that is running at a speed inversely proportional to its quadratic variation, it reduces to a Brownian motion.

If we apply the time change to the original pricing error process \( (Z_t(\theta)) \), then we may deduce from (12) that

\[
Z_{T_t}(\theta) = A_{T_t}(\theta) + U_{T_t} = A_{T_t}(\theta) + B_t.
\]

Therefore, we may now claim that \( (Z_{T_t}(\theta)) \) becomes the standard Brownian motion if and only if \( \theta = \theta_0 \), due to the DDS theorem. Obviously, the bounded variation component \( (A_{T_t}(\theta)) \), even after time change, vanishes when and only when \( \theta = \theta_0 \). The martingale method by Park (2008) uses this fact and defines the value of \( \theta \) as the minimizer of the criterion function

\[
\Phi(\theta) = \int_{\Theta} \{\Phi_N(x, \theta) - \Phi(x)\}^2 d\Phi(x),
\]

where \( \Phi \) is the distribution function of the \( d \)-dimensional multivariate standard normal random vector. The martingale estimator \( \hat{\theta}_N \) of \( \theta_0 \) is then defined as the minimizer of the criterion function \( Q_N \), i.e.,

\[
\hat{\theta}_N = \arg\min_{\theta \in \Theta} Q_N(\theta).
\]

Footnote 17: The choice of \( \Delta \) is more of an empirical matter, which we will discuss in detail later in our empirical section.
The martingale estimator is therefore a minimum-distance estimator with the Cramer-von Mises (CvM) distance between the empirical distribution of the sample under the unknown parameter values and the distribution under the true parameter values. Park (2008) shows that this type of minimum distance estimator is consistent, and asymptotically normal, under mild regularity conditions. The asymptotic variance of the estimator can be obtained by the usual subsampling method.

To introduce the main idea of the methodology more effectively, we assume thus far that the pricing error process \((Z_t(\theta))\) is observed continuously in time for all \(\theta \in \Theta\). This, of course, is not true in our analysis, as is the case for virtually all other potential applications. The methodology can be easily implemented and all the theoretical results continue to hold for discretely sampled observations, as long as the sampling intervals are sufficiently small relative to the time horizon of the samples. This was shown in Park (2008). For our empirical analysis, we use daily observations over approximately fifty years. The necessary modifications required to deal with discretely observed samples are largely trivial and obvious. To obtain the time change, for instance, we use the realized variance of \((P_t)\),

\[
dP_t = dp_t/p_t - r_f^t dt,
\]

instead of its quadratic variation \(([P_t])\), if \((P_t)\) is observed at intervals of length \(\delta > 0\) over time horizon \([0, T]\) with \(T = n\delta\), where \(n\) is the size of discrete samples.

Finally, we may readily allow for the existence of jump components in our model (10). Indeed, we may easily deal with the presence of discrete jumps in our methodology, simply by discarding the observations of \((P_t)\), \(dP_t = dp_t/p_t - r_f^t dt\), over the random time interval \([T_{(i-1)\Delta}, T_i\Delta]\) that is believed to have jumps. All other procedures in our methodology are valid for the remaining observations. In our empirical studies, we use the Hausman-type test of Barndorff-Nielsen and Shephard (2006) for the detection of jumps for each of the random intervals \([T_{(i-1)\Delta}, T_i\Delta]\), \(i = 1, \ldots, N\). Although it is well-known that the jumps are frequently observed for many intra-day samples, it appears that jumps are rare for the samples of daily or lower frequency observations. We detected some evidence of jumps in our daily observations, but their number is relatively small.

### 3.2 Measuring Volatilities of Macroeconomic Variables

Now we spell out how to extract the volatility processes \((\sigma^c_t)\) and \((\sigma^y_t)\). It is much more challenging than to extract the volatility process \((\sigma^m_t)\), since the observations on their underlying processes are available at relative low frequencies like many other macroeconomic variables. As we explained in the previous subsection, \((\sigma^m_t)\) can be readily measured and estimated by the realized variance of market returns at high frequencies.\(^{18}\) However, the identification and estimation of volatility for the processes that are not observed at high frequencies are not straightforward. In the paper, we directly tackle this issue in the following

\(^{18}\)See, e.g., Barndorff-Nielsen and Shephard (2002) for more discussions on the estimation of volatility processes using high-frequency data.
way. First we let the underlying process \( (X_t) \) follow an Ito-diffusion
\[
\frac{dX_t}{X_t} = \mu_t dt + \sigma_t dB_t,
\]
where \((B_t)\) is the standard Brownian motion, and consider the problem of estimating \( (\sigma_t) \), \( \sigma_t = \sigma_t^c \) or \( \sigma_t^p \), under some realistic assumptions, using discrete samples \( (X_{t_j}) \) of \( (X_t) \). It is assumed in our setup here that the sampling intervals \( t_j - t_{j-1}, \ j = 1, \ldots, m \), are not sufficiently small.

Over the interval \([t_{j-1}, t_j]\), we have
\[
\int_{t_{j-1}}^{t_j} \frac{dX_t}{X_t} = \int_{t_{j-1}}^{t_j} \mu_t dt + \int_{t_{j-1}}^{t_j} \sigma_t dB_t. \tag{14}
\]
For many macroeconomic variables, the values of the level \( X_t \) are relatively much larger than its increment \( X_{t_j} - X_{t_{j-1}} \) in any of the intervals \([t_{j-1}, t_j]\) of frequency such as monthly and quarterly. Therefore, it seems reasonable to approximate \( \int_{t_{j-1}}^{t_j} dX_t/X_t \) by \( (X_{t_j} - X_{t_{j-1}})/X_{t_{j-1}} \), i.e., the growth rate of \( (X_t) \) over the interval \([t_{j-1}, t_j]\), for \( j = 1, \ldots, m \).

Moreover, if we assume the drift term \( (\mu_t) \) is continuous, then there exists \( s_j \in [t_{j-1}, t_j] \) such that \( \mu_{s_j} (t_j - t_{j-1}) = \int_{t_{j-1}}^{t_j} \mu_t dt \) for all \( j = 1, \ldots, m \), by the mean value theorem. If, furthermore, \( (\mu_t) \) varies smoothly over time, then we may approximate \( (\mu_{s_j}) \) by \( (\mu_t) \). This appears to be realistic in our case, so we assume that \( (\mu_t) \) is an exogenous function of time for which these approximations are valid. Given the assumption, the drift term \( (\mu_t) \) can be consistently estimated by the standard nonparametric method applied to (14). We adopted the local linear estimation, using the least squares cross-validation method to obtain the optimal bandwidth parameter. The reader is referred to Li and Racine (2007, p.83) for more details.

We exploit two different approaches to extract the volatility process \( (\sigma_t^2) \). First, we consider
\[
\left( \int_{t_{j-1}}^{t_j} \frac{dX_t}{X_t} - \int_{t_{j-1}}^{t_j} \mu_t dt \right)^2 = \int_{t_{j-1}}^{t_j} \sigma_t^2 dt + \left\{ \left( \int_{t_{j-1}}^{t_j} \sigma_t dB_t \right)^2 - \int_{t_{j-1}}^{t_j} \sigma_t^2 dt \right\}, \tag{15}
\]
the left-hand side of which we may approximate well using discrete observations \( (X_{t_j}) \) of \( (X_t) \) as explained above. Note that
\[
\mathbb{E} \left\{ \left( \int_{t_{j-1}}^{t_j} \sigma_t dB_t \right)^2 - \int_{t_{j-1}}^{t_j} \sigma_t^2 dt \mid \mathcal{F}_{t_{j-1}} \right\} = 0
\]
for \( j = 1, \ldots, m \).

\( ^{19} \text{Note that the approximation error is given by } \int_{t_{j-1}}^{t_j} (X_t - X_{t_{j-1}})/(X_t X_{t_{j-1}}) dX_t \text{ and } (X_t - X_{t_{j-1}})/(X_t X_{t_{j-1}}) \approx 0 \text{ for many macroeconomic variables including those we consider here.} \)
As with the drift term \((\mu_t)\), we may regard the diffusion term \((\sigma_t)\) as an exogenous function of time varying smoothly over intervals \([t_{j-1}, t_j]\) for all \(j = 1, \ldots, m\). In this case, we may approximate in (15)

\[
\int_{t_{j-1}}^{t_j} \sigma_t^2 \, dt = \sigma_{s_j}^2 (t_j - t_{j-1}) \approx \sigma_{s_j}^2 (t_j - t_{j-1}),
\]

where \(s_j \in [t_{j-1}, t_j], j = 1, \ldots, m\), and the volatility process \((\sigma_t)\) can be estimated by the standard nonparametric method such as the local linear estimation. We use this approach to extract the volatility processes \((\sigma^x_t)\) and \((\sigma^y_t)\), again with the optimal choice of bandwidth based on the least squares cross-validation. A potential caveat of this nonparametric method would be that this method may produce overly smooth volatility factors. We discuss more on this point in detail in our empirical section.

Second, we suppose that the volatility process is stochastic with an additional source of randomness. For this approach, we let the volatility process \((\sigma_t)\) be random, but remain to be constant over each of the intervals \([t_{j-1}, t_j], j = 1, \ldots, m\). More specifically, we set

\[
\int_{t_{j-1}}^{t_j} \sigma_t \, dB_t = \sigma_j (B_{t_j} - B_{t_{j-1}})
\]

and \((\sigma^2_j)\) to be driven by the logistic transformation of a latent autoregressive factor \((w_j)\), i.e.,

\[
\sigma^2_j = a + \frac{b}{1 + \exp\{-c(w_j - d)\}}
\]

with \(w_j = \rho w_{j-1} + \varepsilon_j\), where \((\varepsilon_j)\) is assumed to be an i.i.d. sequence of standard normals. Note that (16) is the standard Gaussian volatility model in discrete time. We let \((\varepsilon_j)\) be correlated with the Brownian motion \((B_t)\) to allow for the leverage effect. The model parameters \(a > 0, b > 0, c > 0\) and \(d\) determine the actual volatility function. In particular, \(a\) and \(a + b\) represent the two asymptotic values of volatility, and \(c\) and \(d\) respectively the speed and location of transition.

The volatility model introduced above was developed and investigated recently by Kim, Lee and Park (2008). The model can be regarded as an extension of the usual discrete-time stochastic volatility model, which relies on the autoregressive modeling for the logarithmic transformation of volatility. The former is indeed much more flexible than the latter, and has implications that are much more realistic. The latent factor \((w_j)\) and unknown parameters \(a, b, c\) and \(d\) can be estimated by the density-based Kalman filter, or by the Bayesian method using Gibbs sampling method. The reader is referred to Kim, Lee and Park (2008) for more details about the computation procedure and comparison with other existing discrete-time stochastic volatility models.

### 3.3 Calculating Covariances in Mixed Frequencies

Now that we have the extracted volatilities of the macroeconomic variables in their observation frequency, the calculation of the time changed covariances between the market and the macroeconomic variables is straightforward if we have the volatilities in the same
frequency. However, the consumption volatility is estimated in monthly frequency and the labor income volatility in quarterly. Moreover, the market volatility $\sigma_m^\text{m}$ is not estimated yet in any frequency. In this section, we describe a nonparametric interpolation method for the lower frequency macroeconomic volatilities into daily frequency, as well as, a simple way to calculate the market volatility in daily level.

In order to interpolate the daily level of volatilities from lower frequency data, we assume that the volatilities are characterized by a nonparametric function of time. Furthermore, we assume that the extracted volatilities are realizations of the time-varying volatility functions at the mid-point of the observation interval. For instance, if the observation interval is monthly, we assume that the extracted volatilities are realizations of the volatility function at the fifteenth day of each month\textsuperscript{20}. Based on the assumptions, the daily level of volatilities can be calculated by plugging in the time levels which correspond to the daily frequency. For instance, the corresponding daily time level for monthly interval is obtained by finding the grid infilling the monthly time interval at daily frequency. When applying the nonparametric method, we also use the local linear kernel with the smoothing parameter obtained from the least squares cross validation.

For the market volatility, we use the same idea. The difference from the macroeconomic volatilities is that the market volatility is observed in random time interval $[T_{(i-1)\Delta}, T_{i\Delta}]$. In this case, we apply the nonparametric interpolation by assuming that the estimated market volatility $\sqrt{([P]^2_{T_{(i-1)\Delta}} - [P]^2_{T_{i\Delta}})/(T_{i\Delta} - T_{(i-1)\Delta})}$ is a realization at the time $t = (T_{(i-1)\Delta} + T_{i\Delta})/2$. Finally, if the daily volatilities are obtained, we use the Riemann sum to calculate the time changed covariances.

4 Empirical Procedure and Measurement

4.1 Data

We use S&P500 index to calculate the market returns. The index is daily close price adjusted for the dividends and splits, and the returns are obtained by calculating the arithmetic returns of the daily close price. Once we have the daily series of the market returns, we calculate the daily excess returns of the market over the risk free rate of return. For the risk free rate of return, three months treasury bill rates are used. The three months rates are adjusted to the daily level by dividing by 360. Since the daily series on the three months treasury bill rates can be considered as a risk free return from today to tomorrow, the daily excess return on the market portfolio is calculated by subtracting yesterday’s treasury bill rate from today’s return on the market.

We exclude the returns over weekends from our data set because the returns from Friday to Monday seem to have different distribution than the returns on the other day of the week. Especially, the returns on Mondays are significantly negative. This, so called the “Monday effect” or “weekends effect”, has been widely investigated in the literature; see, for

\textsuperscript{20}This mid-point assumption is not necessary and an alternative point in the observation interval can be used, however it is almost impossible to identify the exact point where the mean value is reached from the lower frequency data.
instance, French (1980), Lakonishok and Levi (1982) and Wang et. al. (1997) among others. Settlement effect and clearing delays, or expiration of stock options can be considered as one of the possible explanations for the Monday effect, however it seems hard to include the daily seasonal effect in the asset pricing models that we considered in section 2. Moreover, this may have an important implication related to measuring ambiguity aversion. Even if Monday effect is less likely to be related to ambiguity, the measured conditional volatilities from stock returns may increase due to Monday effect. Thus, without an explicit model handling the daily seasonal effect, there can be a bias of ambiguity aversion estimate in our case\(^{21}\). One might use a dummy variable for Mondays and proceed the analysis (e.g. Fortune (1999)), but in this paper we simply discard the Monday returns.

For the volatilities of macroeconomic variables, such as consumption and labor income, we use monthly real per capita consumption of nondurables plus services and quarterly real per capita labor income. The labor income is defined as wages and salaries plus transfer payments plus other labor income minus personal contributions for social insurance minus taxes, which is used in Lettau and Ludvigson (2001).

The S&P 500 index is obtained from Yahoo finance web site, and the consumption data is obtained from the Bureau of Economic Analysis. The three month treasury bill rate is from the Federal Reserve Bank of St. Louis, and the labor income from Martin Lettau’s web site. The data set covers from January 1, 1960 to December 29, 2006. We report summary statistics on the data set in Table 1.

### 4.2 Implementation of Time Change

Our martingale estimation framework enables us to observe the market returns in the volatility time, not in the usual calendar time, by incorporating the time change. In order to calculate the time change \(T_i\Delta\) with \(i = 1, \ldots, N\), one needs to preset a constant volatility length \(\Delta\) which determines the degree on how often the data should be observed in terms of the volatility time. Given the finite number of observations, it is easy to deduce that a higher volatility length would imply a lower number of samples after time change and vice versa. Common sense will choose the smallest \(\Delta\) to obtain the largest number of samples. This is because usual estimators are more efficient as the number of samples gets larger. Adopting this idea, we find the volatility length \(\Delta\) which is the smallest among all the admissible values of \(\Delta\). Note that the admissible range of \(\Delta\) is determined by a number of factors that are difficult to evaluate in practice. In general, extremely small values of \(\Delta\) can harm the effectiveness of the time change. For instance, if \(\Delta\) is too small, then from the definition of the time change, \([T_{(i-1)\Delta}, T_i\Delta]\) often becomes the same as the observation interval of the data, and therefore the time changed data will have similar property as the original data.

In order to find the admissible range of \(\Delta\), we use the Cramer-von Mises (or CvM) distance for the time changed market returns \(\left\{\int_{T_{(i-1)\Delta}}^{T_i\Delta} \left( dp_t / p_t - r^f t \right) dt \right\} / \sqrt{\Delta}\). Especially, we find the time changes based on different volatility length \(\Delta_k\), with \(k = 1, \ldots, K\), and

\(^{21}\)When we estimate our model with the sample without treating the Monday effect, this indeed happens. Ambiguity aversion appears to become significantly bigger.
calculate the CvM distance for each $\Delta_k$. Since the CvM distance can be interpreted as the degree on how far the empirical distribution of time changed market return departs from $N(0, 1)$, it is useful to check whether time change based on some $\Delta_k$ is effective or not. In other words, if it is close to 0, then the time change based on $\Delta$ works effectively, while if it is far from 0, then the $\Delta$ is supposedly too small or too large to produce the effective time change. Figure 1 shows the CvM distance for $k = 5, 10, \ldots, K$ with $\Delta_k = |P|_{\tau_k}/K$. Note that $k$ is the number of days to be considered to calculate the average quadratic variation and $K$ is the total number of days in the dataset. We can see that the CvM distance drastically decreases as $k$ increases from 5 (or 5-day average quadratic variation) to 20 and then it stabilizes around the level of 0.1 - 0.3. This implies that the time change will work effectively if $k$ is greater or equal to 20. From this point, we set the volatility length as $\Delta_{20}$ or 20-day average quadratic variation.

It is important to note that for the actual calculation of the time change ($T_{i\Delta}$) for $i = 1, \ldots, N$ with the discrete observations, we use the minimum distance criteria for the incremental quadratic variation process. Especially, our time change for the discrete observation is defined as

$$T_{i\Delta} = \arg \min_{s \geq T_{(i-1)\Delta}} \left| [P]_s^\delta - [P]_{T_{(i-1)\Delta}}^\delta - \Delta \right|.$$ 

Our modified time change is different from the original time change in (13) in the sense that it finds the time when the quadratic variation from the previous time change $T_{(i-1)\Delta}$ is closest to the volatility length $\Delta$, while the original time change finds the smallest time when the quadratic variation from initial time is greater than $i\Delta$. Note that this modified time change is not stopping time because the time with minimum distance criteria is unknown based on current information. Nevertheless, for practical purposes, it is advantageous to use the modified time change with discrete observations because it ensures that the quadratic variation on the interval $[T_{(i-1)\Delta}, T_{i\Delta}]$ is closest to $\Delta$ in the given sampling frequency. We use the modified time change in calculating the signature plot mentioned above, as well as in the following empirical analysis.

Once we find the time change ($T_{i\Delta}$) for $i = 1, \ldots, N$, we need to calculate the time changed market return, time changed risk free return, time changed covariance between market and consumption, time changed market variance, time changed covariance between market and labor income, and time changed market volatility. Basically, all the terms in (10) will be calculated and plugged in before estimating the preference parameters ($\alpha, \beta, \kappa$).

Firstly, the time changed market return ($\int dp_t/p_t)^{22}$ and risk free return ($\int r_f^t dt$) is easily obtained by calculating the cumulative return for the intervals $[T_{(i-1)\Delta}, T_{i\Delta}]$ based on the daily returns. Secondly, the time changed variance of the market ($\int (\sigma_m^t)^2 dt$) is calculated from the realized volatility for the interval, i.e., $[P]_{T_{i\Delta}}^\delta - [P]_{T_{(i-1)\Delta}}^\delta$. Thirdly, the macroeconomic volatilities ($\sigma_c^t, \sigma_y^t$) are calculated from two different approaches discussed in section 3.2.

\footnote{For simplicity, we put $\int$ as the integral on the interval $[T_{(i-1)\Delta}, T_{i\Delta}]$}
4.3 Macroeconomic Volatilities

Following the econometric methodology developed in the previous section, we extract the volatilities of macroeconomic variables by two different approaches; (i) time-varying volatility, and (ii) nonlinear stochastic volatility.

Table 2 presents the estimation results of the consumption and labor income based on a Gibbs sampling. We first obtain the demeaned rate of returns for each process from the standard local linear method and exclude the outliers which is greater than 3 in absolute value after normalization. The prior distributions for the Gibbs sampling are based on Kim, Lee, and Park (2008) while the location and scale parameters for $a$, $b$, and $c$ are chosen to be similar to the point estimates and their standard deviations of the maximum likelihood estimation\(^{23}\). For the consumption volatility, the lower and upper bound is estimated to be 0.001 and 0.333, which implies 0.1% of lower bound and 2% of upper bound in annual level. The speed parameter ($c$) is estimated to be 0.089, which implies that the transition from low to high volatility is relatively slow. The leverage effect is estimated to be positive and significant. On the other hand, the labor income volatility has relatively smaller level of lower bound 0.001 (or 0.06% in annual level), and similar level of upper bound 1.076 (or 2.03% in annual level). The speed parameter is estimated to be 1.105, which is much greater than the consumption volatility. This high speed implies that the shift between low and high volatility is fast, or equivalently, the volatility process is closer to a process switching two regimes. The leverage effect is small with 0.05, however it is not significant. The persistency parameter for the latent factor is estimated to be 0.727, which implies that the latent process is highly stationary\(^{24}\).

Figure 2 (a) plots the extracted consumption volatilities from the two approaches and the realized volatility. For the realized volatility, we use the squared monthly consumption growth rate. The growth rate is demeaned by the similar local linear method. We can see that both volatilities from the local linear and Gibbs sampling are decreasing over the sampling period, which coincides with the overall pattern of the realized volatility. Figure 2 (b) presents the Gibbs sampling result compared with the local linear kernel result. In general, the long-run trend of both volatility processes are very similar, however the short-run fluctuations in the Gibbs sampling does not exist in the local linear kernel result. This is well expected from the two different approaches because the stochastic volatility model generalizes the time-varying volatility model by adding additional shocks to the volatility process.

Figure 3 (a) plots the extracted and realized volatilities for the labor income. Compared to the consumption volatility, the labor income volatility seems to move between two different states depending upon the business cycle. This feature coincides with the parameter estimates in Table 2, which reports the smaller lower bound and much higher speed parameter compared to the consumption volatility. If we compare the extracted volatilities from the two different approaches (in Figure 3 (b)), we can see that they almost share the

\(^{23}\)The results of the maximum likelihood estimation is not presented in this paper. However, the results can be provided upon request.

\(^{24}\)For the consumption volatility, we restrict $I(1)$ latent factor. If we estimate the persistency parameter, then it is estimated to be 0.998, which is very close to 1.
long-run trend, while the short-run movement of the Gibbs sampling result suggests that the local linear method produce much smoother estimates of conditional volatilities.

5 Ambiguity, Risk and Intertemporal Substitution

Finally, we are led to discuss our empirical results on the ambiguity aversion, risk aversion, and intertemporal substitutability. We first analyze the baseline case in which financial market is assumed to represent the aggregate wealth. Then, we report our main results and investigate the model (10) in detail.

5.1 Baseline Case: Financial Wealth Only

Table 3 presents estimation results of three configurations for the recursive utility models; power utility (Model I), stochastic differential utility (Model II), and the stochastic differential utility with ambiguity aversion (Model III). In all three settings, it is assumed that financial wealth proxies the total wealth, i.e., \( \pi = 1 \), is imposed. As mentioned earlier, financial wealth is only a subset of the aggregate wealth and therefore we may miss important interactions between human wealth and asset returns in this case. However, to better understand the effect of human wealth, we believe that it is necessary to compare the results from the specifications with and without human wealth. In this light, we set this as our baseline case. Specifically, Model I is used to verify if the equity premium puzzle arises in our setting and data set. The results from Model II are directly comparable to Epstein and Zin (1991), Baskshi and Naka (1997), and Normandin and St-Amour (1998) in that they also used financial wealth as the proxy for the aggregate wealth. To the best of our knowledge, Model III which is our main model has not been empirically studied.

In all three settings, we need to estimate volatilities of the consumption growth \( (\sigma_c^2) \), which we constructed using the method developed in a previous section. To compute conditional covariances, we assume that the correlation \( \rho_c \) between consumption growth and the market return is constant at 0.2. We obtained this value by computing the sample correlation between the two variables and it is consistent with the existing studies.

Result from Model I states that the famous 'equity premium puzzle à la Mehra and Prescott does prevail, showing roughly 258 as the estimate of the relative risk aversion (RRA). Increasing the correlation coefficient to a counterfactual value of 1 still generates an estimate of RRA around 52, confirming that the main reason for the puzzle is the smooth consumption growth. In this case, the elasticity of intertemporal substitution (EIS) is given by the reciprocal of relative risk aversion, and it is estimated to be close to 0. This, in fact, is consistent with the existing studies estimating the EIS such as Hall (1988). In those studies, they use the consumption growth as the regressed and asset returns especially, Treasury bills as a regressor to estimate a linearized Euler equation with homoskedasticity.25 There

\[25\text{However, even in the linearized setup, results are mixed. For instance, Attanasio and Weber (1989) estimated the EIS around 2. Vissing-Jorgensen (2002) reported that the EIS is close to 3 for Treasury returns with bond holders’ consumption and higher than 1 for stock holders. Moreover, if one estimates the linearized equation consisting of consumption growth and asset returns, switching the regressor and the regressed, a high EIS is obtained with asset returns as the regressed, while a low EIS is estimated when} \]
are also numerous studies including Hansen and Singleton (1983), estimating a non-linear Euler equation. Similar to the case of the linearized setup, some studies have reported the EIS close to zero, while others reported significantly positive numbers often greater than one. The empirical literature ascribed this mainly to weak instruments and this basically reveals the difficulty of identifying the key preference parameter. It may also result from the tight restriction imposed on the power utility function and the counterfactual assumption or treatment on the nature of the volatilities of asset returns. With our newly developed econometric tools in hand, we believe that we can handle most of these issues aforementioned. We now examine how alternative models affect the estimation results.

With the stochastic differential utility (Model II), the estimates of the two parameters $\alpha$ and $\beta$ in Model II are $-3.6$ and $0$ respectively. This implies that the estimated risk aversion $(1 - \alpha)$ is dramatically decreasing to 4.6, while the estimated EIS $(1/\beta)$ is reaching a very high number. In comparison with the existing studies in a similar setting, Epstein and Zin (1991) reported the RRA around one and the EIS close to zero via GMM estimations, and Normandin and St-Amour (1998) stated the RRA around 1.4 and the EIS around 1.2 using a maximum likelihood estimation method. Our result suggests that the representative agent is more risk averse and her consumption is highly substitutable across periods compared to the previous results. Although high values of EIS are not incompatible with explaining the behaviors of stock returns or risk-free rates, high standard errors of $\beta$ estimates hint that it requires further investigation. In addition, $\beta$ being close to zero implies that asset returns have little link to consumption growth, which is puzzling. Thus, all of the points lead us to suspecting a weak identification problem for the EIS parameter. As briefly mentioned, a wide range of EIS estimates is not new in the related literature of estimating the preference parameter in both linearized and non-linear Euler equations. However, the literature does not offer much explanation on why it is difficult. Instead, many recent studies have been concentrating on weak instruments to overcome this issue. Although there is little doubt that this is an important task in econometrics, in this paper we focus on understanding the nature of identifying the EIS parameter given that our estimation does not make use of instrumental variables and the estimated models are of highly interpretable forms in continuous time.

First of all, although Model II is an extension of Model I by adding the conditional return variance, the two models have very different implications for linking asset returns to conditional covariances of consumption growth and returns. In case of Model I, small volatility of consumption growth without an additional explanatory variable implies a large coefficient (i.e. a small EIS) to match the average market risk premium. Meanwhile, Model II has an additional explanatory variable and the coefficients of both explanatory variables have non-linear parametric restrictions. For expositional ease, we rewrite the Model II in below:

$$
\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta \alpha}{1 - \beta^2} \rho_c \sigma^c_t \sigma^m_t dt + \left(1 - \frac{\alpha}{1 - \beta}\right) (\sigma^m_t)^2 dt + \sigma^m_t dW_t.
$$

consumption growth is the regressed.

Although some studies such as Yogo (2004), adopt a more flexible preferences relaxing this restriction, due to their linearization and homoscedasticity assumption, they virtually estimate only one parameter.
Between two explanatory variables, it is well known that the conditional variance of market return \((\sigma^m_t)^2\) is more volatile than the conditional covariance between consumption growth and the market return \((\rho_c \sigma_c \sigma^m_t)\). Then, it is possible that there exists some statistical tension for estimating two parameters, because of the relatively weak signal from the consumption growth, provided that both variables have non-trivial correlations with asset returns. Especially, since the EIS is closely related to shifting consumptions across periods without uncertainty, identification of the EIS can be a more challenging task. That is, the equity premium puzzle in the context of the power utility becomes a weak identification problem when we break the tight restriction between risk aversion and intertemporal substitutability. In addition to this, note that the estimated EIS is measured by the reciprocal of \(\beta\). A small perturbation of \(\beta\) coefficient can lead to a large swing of the EIS. For instance, \(\beta = 0.1\) implies the EIS of 10, while \(\beta = 2\) means 0.5. Even if it appears to be a minor issue, this can amplify the weak identification problem given the smooth variations of the consumption volatilities. Therefore, all these factors contribute to a weak identification problem of the EIS parameter. To further analyze this issue, we draw the surfaces and contours of the CvM measures of the model II. The left panels of Figure 4 show that estimating the risk aversion appears to be easily attained and relatively accurate, while it suggests that the elasticity of intertemporal substitution is not going to be easy to estimate due to its flat surface. The left panels in Figure 5 corroborate our conjecture. A clear pattern from the contours is that \(\beta\), the reciprocal of the EIS is small and close to zero and has a very flat region ranging from 0 to 2. Thus, it is not surprising to observe such distant values of the EIS estimates in the literature.

Next we incorporate the ambiguity aversion into the stochastic differential utility setup (Model III). The result clearly dictates that ambiguity aversion is significant both economically and statistically. Specifically, the last column in the Table 3 shows that the estimated RRA is around 0.58-0.74, the EIS is estimated around 0.68-0.78, and the ambiguity aversion parameter \((\kappa)\) is estimated around 0.36. \(\kappa\) measures how much the representative household distorts her beliefs to prepare against a worst case scenario given the ignorance of the true conditional probability distribution. Recall that the conventional notion of the market price of risk measures the degree to which an investor will adjust her probability to be risk neutral. Thus, \(\kappa\) quantifies a constant adjustment of probability in order to be neutral against a Knightian sense of uncertainty. Although it is true that a more sophisticated model of ambiguity aversion such as time-varying ambiguity aversion or learning with ambiguity aversion would further clarify the nature of this new source of premium, our empirical results state that modeling uncertainty differentiated from the usual sense of risk is a first-order business to understand the behaviors of asset returns. Given that, the lower RRA estimates in Model III is understandable because ambiguity aversion captured by the conditional volatility in our setup is likely to alleviate the burden of the return variance in accounting for the average return behaviors. One caveat of the RRA estimate is that its value is somewhat too low and its measurement is noisy. We suspect that the Model III over-compensates the contributions from the market factors compared to the consumption factor. We believe that this results from not including human wealth to construct the aggregate wealth which is the right measure as shown in the model section. Related but not expected, it appears that ambiguity aversion helps identify the EIS as well. Admittedly,
it is still statistically insignificant. But our numerous robustness checks in various dimensions suggest that the inclusion of ambiguity aversion provides the other two explanatory variables, the consumption growth and the rate of return from wealth, with fair chances of explaining asset returns by correctly specifying the existence of ambiguity aversion.

5.2 Human Wealth

Now we state our main results from our continuous-time recursive utility model with human wealth, (10).

\[
\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta \alpha}{1 - \beta} \rho \sigma c \sigma m dt + \left(1 - \frac{\alpha}{1 - \beta}\right) (\sigma m)^2 \pi dt + \left(1 - \frac{\alpha}{1 - \beta}\right) \rho y \sigma y \sigma m (1 - \pi) dt + \kappa \sigma m^2 dt + \sigma m dW_t.
\]

This involves fixing two more parameters \(\pi\) and \(\rho y\), the fraction of financial wealth, and the correlation between labor income growth and the market return. For the former, we tried two values (1/3 and 2/3). Our robustness checks reveal that different values of \(\pi\) give similar results to either of the closer chosen values.\(^{27}\) Regarding the value of \(\rho y\), there is little consensus about it. Several empirical studies report that this correlation is positive, while other studies based on structural models such as Lustig and Van Nieuwerburgh (2006), and Chen et al. (2008) report a strong negative correlation such as \(-0.7\) between asset returns and the imputed human wealth returns. According to our computation it was 0.03. We tried different values such as 0.03 and \(-0.03\), and the results are reported in Table 4.

The main question to be addressed is the effect of including labor income growth. Regarding ambiguity aversion, the estimates of ambiguity aversion rarely vary across settings and the estimates of \(\kappa\) are again around 0.36. Note that this result measured together with the key sources of aggregate risk factors such as the consumption growth, market returns, and labor income growth. Our empirical results strongly suggest that investors will readily take uncertain bets in financial markets only if a sufficient amount of premium is given, separate from the conventional risk premium. Meanwhile, a major difference of the Table 4 in comparison with Table 3 is that the risk aversion coefficient increases. In case of Model II counterparts (i.e., models without ambiguity), the RRA increases from 4.6 up to 14. With ambiguity aversion, the RRA increases from 1.4 up to 5.5. With the addition of the labor income growth, the total wealth becomes less volatile in comparison with the financial wealth. Thus, in order to make up for the level of variability related to the aggregate wealth return term, a higher risk aversion is needed.

Regarding the intertemporal substitutability, the EIS estimates increase like the case of risk aversion when ambiguity aversion is imposed. With \(\pi = 1/3\) and \(\rho y = 0.03\), we have 9.33 with the non-parametric volatility model, and 3.69 with the non-linear stochastic volatility model. With \(\rho y = -0.03\) instead, the estimated EIS is 21.79 with the parametric volatility and 13.22 with the non-linear volatility, but the standard errors are very big.\(^{27}\) We also tried estimating this parameter directly and the estimated values are around 0.2-0.3 in some cases. But due to the weak identification problem, its identification is affected by alternative model settings.
When the fraction of financial wealth $\pi$ is set to $2/3$, the estimated EIS is around 1.3-5.4 depending on settings. Without ambiguity aversion, $\beta$ is again close to zero and therefore, identification is fairly weak. Interestingly, when we impose a strong negative number for $\rho_y$ following the related literature, we have somewhat lower EIS around 1.2 for most cases. However, in all of the settings we have tried, the point estimates of the EIS is higher than one, meaning that economic agents will change their consumptions rather elastically when real interest rate changes. It should be also noted that all the results in Table 3 show that agents prefer early resolution of uncertainty whether or not there exists ambiguity aversion. This makes economic agents unhappy about fluctuations in future utilities, often called the long-run risk channel. For more details on the mechanisms, see Bansal and Yaron (2004), Hansen, Heaton and Li (2008), and Kim et al. (2008) on stock returns. For the term structure modeling along this line, see Piazzesi and Schneider (2006), and Gallmeyer, Kim, and Gonzales (2008).

In a summary, the recursive utility models with both financial and human wealth give most reasonable results when ambiguity aversion is included and the estimates of ambiguity aversion do not depend on alternative setting. Although the estimates of the risk aversion increase as human capital is added, those are still in an acceptable range of values. The weak identification problem of the EIS is also a prevalent feature across different model specifications.

Before we conclude this section, we discuss the relevance of our human capital setup in which the volatility of returns from human capital is measured by that of labor income growth. Returns from human capital are likely to include an additional component which is not fully internalized via labor income growth due to learning-by-doing or interactions across workers. If this unobservable term happens to have a strong time-varying volatilities, our setup needs an extension in this direction. On the other hand, if this additional component has a constant conditional volatility, then this has an interesting implication for our ambiguity aversion. Suppose that the covariations between returns from human wealth and the financial market return are decomposed into a covariation term related to labor income growth and the other from the externality factor, say $Z$, or

$$\rho_h \sigma_t^h \sigma_t^m = \rho_y \sigma_t^y \sigma_t^m + \rho_z \sigma_z \sigma_t^m,$$

where $\rho_z$ and $\sigma_z$ are the correlation between the externality factor and the market return, and the constant conditional volatility of $z$ respectively. Then, (Model III) is extended as

$$\frac{dp_t}{p_t} - r_t^f dt = \frac{\beta \alpha}{1 - \beta} \rho_c \sigma_c \sigma_t^m dt + \left(1 - \frac{\alpha}{1 - \beta} \right) (\sigma_t^m)^2 \pi dt$$

$$+ \left(1 - \frac{\alpha}{1 - \beta} \right) \rho_y \sigma_t^y \sigma_t^m (1 - \pi) dt + \Upsilon \sigma_t^m dt + \sigma_t^m dW_t,$$

where $\Upsilon = \left(1 - \frac{\alpha}{1 - \beta} \right) \rho_z \sigma_z + \kappa$. That is, if this setup were a better approximation of reality, our estimate of ambiguity aversion may include an additional term related to the interaction between human capital and asset returns. Note, however, that the size of $\Upsilon$ depends on

---

28 According to Mehra and Prescott, the acceptable range of the relative risk aversion does not exceed 10.
the sign of $\rho_z$. As mentioned in the beginning of this section, the returns on human capital is strongly negatively correlated with the market return according to the recent empirical research, while the correlation between labor income growth and market returns tends to be positive. If this is the case, $\rho_z < 0$ must hold. That is, given $1 > \alpha/(1 - \beta)$, this implies that estimated ambiguity aversion is even downward biased. Of course a more elaborate quantitative assessment is necessary, but based on all the results, we believe that our estimates on ambiguity aversion are robust and conservative.

5.3 Ambiguity and Return Volatility: A Further Investigation

The asset pricing equation (11) serves its purpose well to identify the key preference parameters and our empirical results strongly suggest that return volatilities matter in understanding equity premium for holding financial assets. Since asset prices are determined in equilibrium, their return volatilities must be explained by the volatilities of fundamental macro shock processes. Given our theoretical setting, it must be the case that ambiguity about fundamental shock processes amplifies small but persistent macro volatilities such that high return volatilities are generated as observed in data. To theoretically tackle this issue, we assume specific forms of stochastic processes as follows:

$$
\frac{dC_t}{C_t} = \xi_t dt + \sqrt{x_t} \sigma_c dW_{ct},
$$

$$
d\xi_t = (\eta_0 - \eta_1 \xi_t) dt + \sqrt{x_t} \sigma_{\xi} dW_{\xi t},
$$

$$
dx_t = (\eta_0^x - \eta_1^x x_t) dt + \sqrt{x_t} \sigma_{x} dW_{x t},
$$

where $\text{corr}(dW_{it}, dW_{jt}) = \rho_{ij} dt$. $\xi_t$ is the conditional mean of consumption growth rate and $x_t$ is the common volatility factor. Then, we can setup a Hamilton-Jacobi-Bellman equation of the optimization problem (4) as

$$
0 = C_t^{1-\beta} \frac{\alpha + \beta - 1}{1 - \beta} J^{\frac{\alpha + \beta - 1}{\alpha}} - \frac{\phi \alpha}{(1 - \beta)} \frac{\sigma_c^2}{J} + \frac{J_{\xi}}{J} (\eta_0 - \eta_1 \xi_t) + \frac{J_x}{J} (\eta_0^x - \eta_1^x x_t) + \frac{J_g}{J} \mu_{gt} + \frac{1}{2} \left( \frac{J_{\xi}^2}{J} \sigma_{\xi}^2 \xi_t + 2 \frac{J_{\xi} x}{J} \rho_{\xi x} \sigma_{\xi} \sigma_x x_t + 2 \frac{J_{\xi}}{J} \sigma_{\xi} \sqrt{x_t} \sigma_{\xi} + \frac{J_x^2}{J} \sigma_x^2 x_t + \frac{J_g}{J} \sigma_{gt}^2 \right),
$$

with the dynamic budget constraint

$$
dG_t = \mu_{gt} dt + \sigma_{gt} dW_t
$$

where $\mu_{gt} = (G_t - C_t) \theta_t \cdot R_t$, $\sigma_{gt} = \theta'^t \sigma_{Rt}$, $J = J(x, \xi, G)$ is the solution to the above PDE and $J_i$ or $J_{ij}$ is the partial differentials. If we denote conditional mean and volatility of equity premium by $\mu_{Rt}$ and $\sigma_{Rt}$, we can show in the appendix XX the following results by solving an approximate PDE of the above:

$$
\sigma_{Rt} = \left[ \sigma_c + \frac{(1 - \beta) \sigma_{\xi}}{h_1 + \eta_1} + \frac{(1 - \beta)}{\alpha \beta} A_x \sigma_x \right] \sqrt{x_t},
$$

$$
\mu_{Rt} = (1 - \alpha) \sigma_{Rt} \sigma_{Rt}' + \frac{\alpha \beta \sigma_{\xi} \sigma_{Rt}'}{h_1 + \eta_1} - \sigma_{xt} \sigma_{Rt} A_x,
$$

(17)
where

\[ A_x = -\Gamma_1 - \sqrt{\Gamma_1^2 - \Gamma_2}, \]

\[ \Gamma_1 = \frac{-\alpha h_1 - \eta_1 + \rho_{cc\xi} \xi \beta + \alpha \rho_{cc\xi} \xi (h_1 + \eta_1)^{-1} - \frac{\kappa \sigma^2}{\alpha \sigma^2_x}}{(\sigma_x)^2}, \]

\[ \Gamma_2 = \left( \frac{\beta}{\sigma_x} \right)^2 \left[ (A_x \xi \sigma_x)^2 + \alpha (\alpha - 1) \sigma_x^2 + \frac{2 \alpha^2 \rho_{cc\xi} \xi (h_1 + \eta_1)}{h_1 + \eta_1} - \frac{\kappa}{\sqrt{x}} \left( \alpha \sigma_c + \frac{\sigma_x}{h_1 + \eta_1} \right) \right], \]

with \( h_1 \) being the steady state value of \( C/G \) and \( \bar{x} \) denoting a value around which \( \sqrt{x} \) is approximated. (17) states that return volatilities come from three sources; consumption growth, time-varying volatilities of consumption growth, time-varying means of consumption growth. That is, it is the stochastic nature reflected in the conditional mean and volatility of the fundamental macro shock that can increase return volatilities. Note that the ambiguity aversion \( (\kappa > 0) \) increases return volatility via \( (1 - \beta) A_x / \alpha \beta \) term when \( 1 > \beta, \alpha < 0, \) and \( A_x < 0 \). The first two conditions are confirmed in our estimations and the third one is likely to hold when some calibrated parameters are plugged in. Our main interest lies in the last term of \( \Gamma_1 \) which dictates that time-varying uncertainty \( (\sigma^2_x) \) will increase return volatility more if ambiguity aversion is larger (i.e. a bigger positive value of \( \kappa \)) compared to relative risk aversion \( (-\alpha) \). Put differently, given \( \kappa \), a higher relative risk aversion will reduce the role of ambiguity aversion and vice versa.\(^{29}\) Therefore, our setup sheds light on the volatility puzzle in that return volatilities can be amplified via the interaction between preferences against uncertainty and the time-varying uncertainty in macro shocks. How does this uncertainty feed into asset returns? (18) provides some clues on this question. The first term represents a conventional return-volatility relationship. However, in our framework, the return volatility is endogenously determined as a mixture of risk and uncertainty. The second term is so called the long-run risk channel which has been paid much attention recently. The last term is the volatility channel in which ambiguity aversion plays an important role according to our empirical analysis. [more]

6 Conclusion

We began this paper with a question asking if there is an important role played by decision makers’ fear on ambiguity on true probability measure. The answer to this question is positive based on our empirical analysis.

In terms of economic theory, the inclusion of ambiguity aversion is meaningful because it can overcome the Ellsberg paradox. Multiple-priors utility models were developed to incorporate such ambiguity aversion, and have a neat expression for asset prices available in continuous time. In addition, one can view that multiple-priors models as an extension of the rational expectation in that investors may be of insufficient knowledge about the true

\(^{29}\)The effect from \( \Gamma_2 \) in this regard is somewhat obscure. Since \( \alpha \sigma^2 \) is negative and \( \sigma^2/(h_1 + \eta_1) \) is positive, the effect of ambiguity aversion depends on the sign of the sum of these two terms. If the latter is bigger, then larger \( \kappa \) will further increase return volatility. Given the small consumption volatility, this is more likely to be the case, quantitatively.
probability density. When ambiguity aversion is assumed, economic agents are basically endowed with a set of beliefs on the true probability distribution and choose the one that is the least ambiguous. Of course, whether or not the ambiguity aversion matters is an empirical and quantitative concern. Our estimation results strongly suggest that there exists a premium for bearing market uncertainty separate from the conventional risk sources. Even with various specifications, the preference parameter indicating the ambiguity aversion is both economically and statistically significant. In addition, the estimates of the ambiguity aversion parameter rarely vary across alternative settings.

Another interesting finding is that the models with ambiguity aversion have lower relative risk aversion. Thus, the conjectures in Epstein and Wang (1994) or Chen and Epstein (2002) are confirmed in our empirical result. Empirically speaking, our results suggest that relative risk aversion can be estimated with an upward bias unless ambiguity aversion is properly adjusted. What is even more interesting is that incorporation of ambiguity aversion does not dominate the role of risk aversion. Clearly, there exist some independent dimensions of generating premiums for bearing such risk and uncertainty. It is true that we only employed a simple case of ambiguity aversion. Therefore, it would be interesting to further study empirical links of alternative forms of ambiguity aversion to asset prices.

With regard to the elasticity of substitution, there exists a weak identification problem due to its non-linear parametric restrictions and the weak signal from consumption growth. That said, the models with ambiguity aversion still produce quite reasonable estimates of the intertemporal substitution. Therefore, ambiguity aversion not only matters in terms of explaining the behaviors of asset returns, but also helping identify key preference parameters.

In addition to the empirical findings, another contribution of our paper is that we provide a novel econometric approach estimating and testing for continuous-time asset pricing models including both financial and macroeconomic variables. In the empirical analysis of such models, it has long been a tradition that we ignore the availability of high-frequency observations on financial variables, mostly for the lack of ideas about how to use them constructively. Virtually all empirical studies of such models have been done only using lower-frequencies, at which all involved macroeconomic variables are also available. Our paper makes it clear that this is an important loss of information.

In our analysis, we use the available high-frequency observations directly to identify our model, and also nonparametrically correct for time-varying stochastic volatility in the price equation errors. It is widely known that many asset returns show strong evidence for the presence of time-varying stochastic volatility. Unless properly and carefully taken care of, the time-varying stochastic volatility may well have a fatal effect on our estimation results. We believe that our method can be used in many other interesting applications to unravel the complicated interactions between financial markets and macroeconomy.
References


Table 1: Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>Market</th>
<th>Risk Free</th>
<th>Consumption</th>
<th>Labor Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean*</td>
<td>0.12839</td>
<td>0.05551</td>
<td>0.02168</td>
<td>0.02309</td>
</tr>
<tr>
<td>Std. Dev.*</td>
<td>0.13973</td>
<td>0.00146</td>
<td>0.01243</td>
<td>0.01713</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.25178</td>
<td>1.12173</td>
<td>-0.04598</td>
<td>0.43410</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.96404</td>
<td>4.93917</td>
<td>3.94341</td>
<td>6.65616</td>
</tr>
<tr>
<td>Auto. Coef.</td>
<td>0.04900</td>
<td>0.99910</td>
<td>-0.23070</td>
<td>-0.06390</td>
</tr>
</tbody>
</table>

Note: Summary statistics for market, risk-free, consumption, and labor income. Market is the daily S&P 500 index returns, risk-free is the daily three month treasury bill rate divided by 360, consumption is the monthly growth rate of real per capita non-durable plus service consumption, and labor is the quarterly growth rate of labor income defined in Lettau and Ludvigson (2001). *For the purpose of presentation, we report the annualized mean and standard deviation of the series.
Table 2: Gibbs Sampling Results for Consumption and Labor Income

<table>
<thead>
<tr>
<th></th>
<th>Consumption</th>
<th></th>
<th></th>
<th>Labor Income</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$G(1, 0.001)$</td>
<td>0.001</td>
<td>(0.001)</td>
<td></td>
<td>$G(1, 0.001)$</td>
</tr>
<tr>
<td>$b$</td>
<td>$G(1, 0.2)$</td>
<td>0.332</td>
<td>(0.036)</td>
<td></td>
<td>$G(1, 1.2)$</td>
</tr>
<tr>
<td>$c$</td>
<td>$G(1, 0.2)$</td>
<td>0.089</td>
<td>(0.020)</td>
<td></td>
<td>$G(1, 2)$</td>
</tr>
<tr>
<td>$w_0$</td>
<td>$N(40, 40^2)$</td>
<td>2.617</td>
<td>(3.913)</td>
<td></td>
<td>$N(-4, 4^2)$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$U[-1, 1]$</td>
<td>0.544</td>
<td>(0.219)</td>
<td></td>
<td>$U[-1, 1]$</td>
</tr>
</tbody>
</table>

Note: The table presents the estimation results of nonlinear stochastic volatility model based on a Gibbs sampling. We sample 30000 iterations and discards 15000 iterations. The sample period for consumption is from October 1959 to December 2006 and for labor income first quarter of 1959 to fourth quarter of 2006. $U[\theta_1, \theta_2]$ denotes uniform distribution with a support $(\theta_1, \theta_2)$ and $G(\theta_1, \theta_2)$ denotes gamma distribution with mean $\theta_1 \theta_2$ and variance $\theta_1 \theta_2^2$, and $N(\theta_1, \theta_2)$ denotes normal distribution with mean $\theta_1$ and variance $\theta_2$. Pos. Mean and Pos. Std. Dev. denote the posterior mean and posterior standard deviation, respectively. All the parameters are estimated for the scaled data with 100 (or in percentage level).
Table 3: Estimation Results for Baseline Models

<table>
<thead>
<tr>
<th></th>
<th>Model I</th>
<th>Model II</th>
<th>Model III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Panel A: Time-Varying Volatility</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>257.991 (3.589)</td>
<td>0.000</td>
<td>1.290 (0.856)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-</td>
<td>-3.572 (0.063)</td>
<td>0.263 (0.719)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>-</td>
<td>-</td>
<td>0.360 (0.022)</td>
</tr>
<tr>
<td>RA</td>
<td>257.991 (3.589)</td>
<td>4.572 (0.063)</td>
<td>0.737 (0.719)</td>
</tr>
<tr>
<td>EIS</td>
<td>0.004 (0.000)</td>
<td>$\infty$</td>
<td>0.775 (0.514)</td>
</tr>
<tr>
<td>CvM</td>
<td>0.035</td>
<td>0.034</td>
<td>0.031</td>
</tr>
<tr>
<td></td>
<td>Panel B: Nonlinear Stochastic Volatility</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>258.881 (3.508)</td>
<td>0.000</td>
<td>1.461 (0.812)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-</td>
<td>-3.572 (0.063)</td>
<td>0.419 (0.692)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>-</td>
<td>-</td>
<td>0.361 (0.021)</td>
</tr>
<tr>
<td>RA</td>
<td>258.881 (3.508)</td>
<td>4.572 (0.063)</td>
<td>0.581 (0.692)</td>
</tr>
<tr>
<td>EIS</td>
<td>0.004 (0.000)</td>
<td>$\infty$</td>
<td>0.684 (0.380)</td>
</tr>
<tr>
<td>CvM</td>
<td>0.035</td>
<td>0.034</td>
<td>0.031</td>
</tr>
</tbody>
</table>

Note: The table reports the estimation results for the asset pricing models in which the aggregate wealth consists of only financial wealth. All results are for the sample 1/2/1960-12/29/2006. The first column is Model I with standard additive CRRA utility, the second column is Model II with recursive utility, and the third column is Model III with multiple-priors recursive utility. The correlation between the market return and the consumption growth ($\rho_c$) is set to be 0.2. The standard errors in parenthesis are obtained by the subsampling method.
Table 4: Implications of Human Wealth

<table>
<thead>
<tr>
<th></th>
<th>Without Ambiguity</th>
<th>With Ambiguity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calibrated values</td>
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</tr>
<tr>
<td>( \pi )</td>
<td>0.333</td>
<td>0.333</td>
</tr>
<tr>
<td>( \rho_y )</td>
<td>0.030</td>
<td>-0.030</td>
</tr>
</tbody>
</table>

Panel A: Time-Varying Volatility

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>-12.628</td>
<td>-12.831</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>RA</td>
<td>13.628</td>
<td>13.831</td>
</tr>
<tr>
<td>EIS</td>
<td>(\infty)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>CvM</td>
<td>0.034</td>
<td>0.034</td>
</tr>
</tbody>
</table>

Panel B: Nonlinear Stochastic Volatility

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>-12.635</td>
<td>-12.825</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>-</td>
<td>-</td>
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<tr>
<td>RA</td>
<td>13.635</td>
<td>13.825</td>
</tr>
<tr>
<td>EIS</td>
<td>(\infty)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>CvM</td>
<td>0.034</td>
<td>0.034</td>
</tr>
</tbody>
</table>

Note: The table reports the estimation results for the asset pricing models in which the aggregate wealth consists of financial wealth and human wealth. All results are for the sample 1/2/1960-12/29/2006. In each panel, each column represents the point estimates and their standard errors for the recursive utility model given the proportion of financial wealth to the aggregate wealth (\( \pi \)) and the correlation between the return on human wealth and financial wealth (\( \rho_y \)). The correlation between the market return and the consumption growth (\( \rho_c \)) is set to be 0.2. The standard errors in parenthesis are obtained by the subsampling method.
Figure 1: Signature Plot of CvM Distance for S&P 500 Index Excess Returns

Note: The $x$-axis represents the number of days $k$ included to calculate the volatility length $\Delta_k$, i.e., $\Delta_k = [P]^3_t k / K$, where $[P]^3_t$ is the realized variance of $(P_t)$, $dP_t = dp_t / p_t - r_t dt$, computed using daily observations over the time horizon $[0, T]$, and $K$ is the total number of days. The $y$-axis represents the CvM distance for the standardized excess returns after the time change.
Figure 2: Consumption Volatilities

(a) Comparison between squared growth rate and estimated volatilities

(b) Comparison between time-varying volatility and nonlinear stochastic volatility

Note: In Figure (a), the volatilities of consumption growth rate from 1960 to 2006 are presented against the squared growth rates. The growth rates are demeaned by the local linear method. The bandwidths for the local linear kernel estimation for time-varying mean and volatilities are selected separately, and are based on the least squares cross-validation (see Li and Racine (2007, p. 83)). In Figure (b), the extracted volatility from the nonlinear stochastic volatility model is presented with the result from the time-varying volatility model. The volatility is estimated by a Gibbs sampling method and the extracted volatility is obtained by plugging in the posterior sample means for the parameters and the latent factor.
Figure 3: Labor Income Volatilities

(a) Comparison between squared growth rate and estimated volatilities

(b) Comparison between time-varying volatility and nonlinear stochastic volatility

Note: In Figure (a), the volatilities of labor income growth rate from 1960 to 2006 are presented against the squared growth rates. The growth rates are demeaned by the local linear method. The bandwidths for the local linear kernel estimation for time-varying mean and volatilities are selected separately, and are based on the least squares cross-validation (see Li and Racine (2007, p. 83)). In Figure (b), the extracted volatility from the nonlinear stochastic volatility model is presented with the result from the time-varying volatility model. The volatility is estimated by a Gibbs sampling method and the extracted volatility is obtained by plugging in the posterior sample means for the parameters and the latent factor.
Figure 4: Surface Plots of CvM Measure for Model II and Model III

(a) Time-Varying Volatility

(b) Nonlinear Stochastic Volatility

Note: The above figures plot the surfaces of the CvM distance on the parameter vector of $(\beta, -\alpha/(1 - \beta))$. Figure (a) presents the result based on the macroeconomic volatilities from the time-varying volatility model, and Figure (b) presents the result from the nonlinear latent stochastic volatility model. In case of Model III, the surfaces are plotted with $\kappa = 0.360$ (in (a)), and $\kappa = 0.361$ (in (b)).
Figure 5: Contour Plots of CvM Measure for Model II and Model III

(a) Time-Varying Volatility

(b) Nonlinear Stochastic Volatility

Note: The above figures plot the contours of the CvM distance on the parameter vector of $(\beta, -\alpha / (1 - \beta))$. Figure (a) presents the result based on the macroeconomic volatilities from the time-varying volatility model, and Figure (b) presents the result from the nonlinear latent stochastic volatility model. In case of Model III, the contours are plotted with $\kappa = 0.360$ (in (a)), and $\kappa = 0.361$ (in (b)).

Figure 6: Value Plot of CvM Distances for Model III

(a) Time-Varying Volatility

(b) Nonlinear Stochastic Volatility

Note: The above figures presents the values plot of the CvM distance with respect to $\kappa$. Figure (a) plots the values of CvM distance with $(\alpha, \beta) = (1.290, 0.263)$ and Figure (b) with $(\alpha, \beta) = (1.461, 0.419)$.